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THE FIBONACCI NUMBERS OF ASYMPTOTICALLY LACUNARY χ^2 OVER PROBABILISTIC $p-$ METRIC SPACES

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Abstract. In this paper we study the concept of almost asymptotically lacunary statistical χ^2 over probabilistic $p-$ metric spaces and discuss some general topological properties of above sequence spaces.

Keywords: analytic sequence, double sequences, χ^2 space, Fibonacci number, Musielak-modulus function, probabilisitic $p-$ metric space, asymptotically equivalence, statistical convergent.

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1. Introduction

In this paper we study the concept of almost asymptotically lacunary statistical χ^2 over probabilistic $p-$ metric spaces defined by Musielak. Since the study of convergence in PP-spaces is fundamental to probabilistic functional analysis, we feel that the concept of almost asymptotically lacunary statistical χ^2 over probabilistic p-metric spaces defined by Musielak in a PP-space would provide a more general framework for the subject.

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in [4]. Later it was investigated by [3, 7–10, 12, 13, 20, 22–29, 31–43],

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for\, some\, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\}$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

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where $t = (t_{mn})$ be the sequence of strictly positive reals t_{mn} for all $m,n \in \mathbb{N}$ and $p-lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. [6] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and obtained the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, [44] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. ([14]- [19]) and ([31]- [42]) have independently introduced the statistical convergence and Cauchy for double sequences and established the relation between statistical convergent and strongly Cesàro summable double sequences. [1] defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. [2] introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Recently [30] have studied the space $\chi^2_M(p,q,u)$ of double sequences and proved some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [11] as an extension of the definition of strongly Cesàro summable sequences. [5] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [21] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a,b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all \quad finite \quad sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0^{m,n}} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

Let M and Φ be mutually complementary modulus functions. Then, we have

(i) For all $u,y \geq 0$, see [40]

$$uy \leq M(u) + \Phi(y), \tag{2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \tag{4}$$

In [41] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - f_{mn}(u) : u \geq 0\},\, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined by

$$t_f = \left\{ x \in w^2 : I_f\left(|x_{mn}|\right)^{1/m+n} \to 0\, as\, m, n \to \infty \right\},$$

where I_f need not be convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|\right)^{(1/m)+n},\, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn} \left\{ inf \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{(1/m)+n}}{mn}\right) \right) \leq 1 \right\}.$$

The notion of difference sequence spaces (for single sequences) was introduced by [42] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the cases $1 \leq p \leq \infty$ and $0 < p < 1$ [1]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},\, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1}x_{mn} - \Delta^{m-1}x_{mn+1} - \Delta^{m-1}x_{m+1n} + \Delta^{m-1}x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{m+i,n+j}$.

2. Definition and preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \dots, x_n) = \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p = 0$ if and and only if $d_1(x_1, 0), \dots, d_n(x_n, 0)$ are linearly dependent,
(ii) $\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\left((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)\right) = \left(d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p\right)^{1/p} for 1 \leq p < \infty$; (or)
(v) $d\left((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)\right) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the $p-$product metric of the Cartesian product of $n-$metric spaces is the $p-$norm of the n-vector of the norms of the $n-$sub spaces.

A trivial example of $p-$ product metric of $n-$ metric space is the $p-$ norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the $p-$ norm:

$$\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_E = sup\left(|det(d_{mn}\left(x_{mn}, 0\right))|\right) =$$
$$sup\left(\left|\begin{matrix} d_{11}\left(x_{11}, 0\right) & d_{12}\left(x_{12}, 0\right) & \dots & d_{1n}\left(x_{1n}, 0\right) \\ d_{21}\left(x_{21}, 0\right) & d_{22}\left(x_{22}, 0\right) & \dots & d_{2n}\left(x_{1n}, 0\right) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}\left(x_{n1}, 0\right) & d_{n2}\left(x_{n2}, 0\right) & \dots & d_{nn}\left(x_{nn}, 0\right) \end{matrix}\right|\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Definition 2.1. *Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k, \ell-$ th term of Ax is as follows: $(Ax)_{k\ell} = \sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ such transformation is said to be non-negative if $a_{k\ell}^{mn}$ is non-negative.*

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which both added an adiditional assumption of boundedness. This assumption was made since a double sequence which is $P-$ convergent is not necessarily bounded.

Let λ and μ be two sequence spaces and $A = \left(a_{k,\ell}^{mn}\right)$ be a four dimensional infinite matrix of real numbers $\left(a_{k,\ell}^{mn}\right)$, where $m, n, k, \ell \in \mathbb{N}$. Then, we say A defines a matrix mapping from λ into μ and we denoe it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mn}) \in \lambda$ the sequence $Ax = \{(Ax)_{k\ell}\}$, the $A-$ transform of x, is in μ.

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series converges for each $k, \ell \in \mathbb{N}$. A sequence x is said to be $A-$ summable to α if Ax converges to α which is called as the $A-$ limit of x.

Lemma 2.1 (See [11]). *Matrix $A = \left(a_{k,\ell}^{mn}\right)$ is regular if and only if the following three conditions hold:*
(1) There exists $M > 0$ such that for every $k, \ell = 1, 2, \cdots$ the following inequality holds:

$\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}|a_{k\ell}^{mn}| \leq M$;

(2) $\lim\limits_{k,\ell\to\infty} a_{k\ell}^{mn} = 0$ *for every* $k, \ell = 1, 2, \cdots$

(3) $\lim\limits_{k,\ell\to\infty}\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} a_{k\ell}^{mn} = 1.$

Let (q_{mn}) *be a sequence of positive numbers and*

$$Q_{k\ell} = \sum_{m=0}^{k}\sum_{n=0}^{\ell} q_{mn}\,(k,\ell\in\mathbb{N})\,. \tag{5}$$

Then, the matrix $R^q = (r_{k\ell}^{mn})^q$ *of the Riesz mean is given by*

$$(r_{k\ell}^{mn})^q = \begin{cases} \frac{q_{mn}}{Q_{k\ell}} & if\, 0 \leq m,n \leq k,\ell \\ 0 & if (m,n) > k\ell. \end{cases} \tag{6}$$

The fibonacci numbers are the sequence of numbers $f_{k\ell}^{mn}\,(k,\ell,m,n\in\mathbb{N})$ *defined by the linear recurrence equations* $f_{00} = 1$ *and* $f_{11} = 1, f_{mn} = f_{m-1n-1} + f_{m-2n-2}; m,n \geq 2$. *Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, some basic properties of Fibonacci numbers are the following.*

$\sum\limits_{k=1}^{m}\sum\limits_{\ell=1}^{n} f_{mn} = f_{m+2n+2} - 1; m,n \geq 1,$

$\sum\limits_{k=1}^{m}\sum\limits_{\ell=1}^{n} f_{mn}^2 = f_{mn}f_{m+1n+1}; m,n \geq 1,$

$\sum\limits_{k=1}^{\infty}\sum\limits_{\ell=1}^{\infty} \frac{1}{f_{k\ell}^{mn}}$ *converges.*

In this paper, we define the fibonacci matrix $F = (f_{k\ell}^{mn})_{m,n=1}^{\infty}$, *which differs from existing Fibonacci matrix by using Fibonacci numbers* $f_{k\ell}$ *and introduce some new sequence spaces* χ^2 *and* Λ^2. *Now, we define the Fibonacci matrix* $F = (f_{k\ell}^{mn})_{m,n=1}^{\infty}$, *by*

$$(f_{k\ell}^{mn}) = \begin{cases} \frac{f_{k\ell}}{f_{(k+2)(\ell+2)-1}} & if\, 0 \leq k \leq m; 0 \leq \ell \leq n \\ 0 & if (m,n) > k\ell \end{cases}$$

that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0... & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0... & \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0... & \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0... & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is obvious that the four dimensional infinite matrix F *is triangular matrix. Also it follows from lemma 2.2 that the method* F *is regular.*

Definition 2.2. *A function* $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_\circ{}^{+}\times\mathbb{R}_\circ{}^{+}$ *is called a distribution if it is non-decreasing and left continuous with* $inf_{t\in\mathbb{R}\times\mathbb{R}} f(t) = 0$ *and* $\sup\limits_{t\in\mathbb{R}_\circ{}^{+}\times\mathbb{R}_\circ{}^{+}} f(t) = 1$. *We will denote the set of all distribution functions by* D.

Definition 2.3. *A triangular metric, briefly t-over probabilisitic* $p-$ *metric spaces, is a binary operation on* $[0,1]$ *which continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continous mapping* $* : [0,1]\times[0,1] \to [0,1]\times[0,1]$ *such that* $a,b,c \in [0,1]$:

(1) $a * 1 = 1,$

(2) $a * b = b * a$
(3) $c * d \geq a * b$ *if* $c \geq a$ *and* $d \geq b$,
(4) $(a * b) * c = a * (b * c)$.

Definition 2.4. *A triple* $(X, P, *)$ *is called a probabilistic p− metric space or shortly* $PP-$ *space if* X *is a real vector space,* P *is a mapping from* $X \times X \rightarrow D \times D$ *(for* $x \in X$*, the distribution function* $P(x)$ *is denoted by* P_x *and* $P_x(t)$ *is the value of* P_x *at* $t \in \mathbb{R} \times \mathbb{R}$*) and* $*$ *is a* $t - p-$ *metric satisfying the following conditions:*
(i) $P_x(\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p) = 0$ *if and and only if* $P_x(d_1(x_1, 0), \ldots, d_n(x_n, 0))$ *are linearly dependent,*
(ii) $P_x(t\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p) = 1$ *is invariant under permutation,*
(iii) $P_x(\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p) = |\alpha| P_x(\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p), \alpha \in \mathbb{R} \setminus \{0\}$.
(iv) $P_x(d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n))) = P_x\Big((d_X(x_1, x_2, \cdots x_n)^p) + P_x(d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}\Big)$
$for 1 \leq p < \infty$; *(or)*
(v) $P_x(d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n))) := \sup\{P_x(\{d_X(x_1, x_2, \cdots x_n)), P_x(d_Y(y_1, y_2, \cdots y_n)\})\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$.

Definition 2.5. *A triple* $(X, P, *)$ *be a* $PP-$ *space. Then a sequence* $x = (x_{mn})$ *is said to convergent* $\bar{0} \in X$ *with respect to the probabilistic p− metric* P *if, for every* $\epsilon > 0$ *and* $\theta \in (0, 1)$, *there exists a positive integer* $m_0 n_0$ *such that* $P_{x_{mn} - \bar{0}}(\epsilon) > 1 - \theta$ *whenever* $m, n \geq m_0 n_0$. *It is denoted by* $P - lim x = \bar{0}$ *or* $x_{mn} \overset{P}{\rightarrow} \bar{0}$ *as* $m, n \rightarrow \infty$.

Definition 2.6. *A triple* $(X, P, *)$ *be a* $PP-$ *space. Then a sequence* $x = (x_{mn})$ *is called a Cauchy sequence with respect to the probabilistic p− metric* $P-$ *if, for every* $\epsilon > 0$ *and* $\theta \in (0, 1)$ *there exists a positive integer* $m_0 n_0$ *such that* $P_{x_{mn} - x_{rs}}(\epsilon) > 1 - \theta$ *for all* $m, n > m_0 n_0$ *and* $r, s > r_0 s_0$.

Definition 2.7. *A triple* $(X, P, *)$ *be a* $PP-$ *space. Then a sequence* $x = (x_{mn})$ *is said to analytic in* X*, if there is a* $u \in \mathbb{R}$ *such that* $P_{x_{mn}}(u) > 1 - \theta$*, where* $\theta \in (0, 1)$. *We denote by* Λ^{2P} *the space of all analytic sequences in* $PP-$ *space.*

Definition 2.8. *Two non-negative sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are asymptotically equivalent* $\bar{0}$ *if*

$$\lim_{m,n} \frac{x_{mn}}{y_{mn}} = \bar{0}$$

and it is denoted by $x \equiv \bar{0}$.

Definition 2.9. *Let* K *be the subset of* $\mathbb{N}$*, the set of natural numbers. Then the asymptotically density of* K*, denoted by* $\delta(K)$, *is defined as*

$$\delta(K) = \lim_{k,\ell} \tfrac{1}{k\ell} |\{m, n \leq k, \ell : m, n \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 2.10. *A number sequence* $x = (x_{mn})$ *is said to be statistically convergent to the number* $\bar{0}$ *if for each* $\epsilon > 0$*, the set* $K(\epsilon) = \Big\{m \leq k, n \leq \ell : (m+n)! |x_{mn} - \bar{0}|^{1/m+n} \geq \epsilon\Big\}$ *has asympototic density zero*

$$\lim_{k\ell} \tfrac{1}{k\ell} \Big|\Big\{m \leq k, n \leq \ell : ((m+n)! |x_{mn} - \bar{0}|)^{1/m+n} \geq \epsilon\Big\}\Big| = 0.$$

In this case we write $St - lim x = \bar{0}$.

Definition 2.11. *The two non-negative double sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are said to be asymptotically double equivalent of multiple* L *provided that for every* $\epsilon > 0$,

$$\lim_{k,\ell} \tfrac{1}{k,\ell} \left| \left\{ (m,n) : m \leq k, n \leq \ell, \left| \tfrac{x_{mn}}{y_{mn}} - L \right| \geq \epsilon \right\} \right| = 0.$$

and simply asymptotically double statistical equivalent if $L = 1$. *Furthermore, let* $S^{L}_{\theta_{rs}}$ *denote the set of all sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *such that* $x\, is\, equivalent to\, y$.

Definition 2.12. *Let* $\theta_{rs} = \{(m_r, n_s)\}$ *be a double lacunary sequence; the two double sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are said to be asymptotically double lacunary statistical equivalent of multiple L provided that for every* $\epsilon > 0$,

$$\lim_{r,s} \tfrac{1}{h_{r,s}} \left| \left\{ (m,n) \in I_{r,s} : \left| \tfrac{x_{mn}}{y_{mn}} - L \right| \geq \epsilon \right\} \right| = 0$$

and simply asymptotically double lacunary statistical equivalent if $L = 1$.*Furthermore, let* $S^{L}_{\theta_{rs}}$ *denote the set of all sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *such that* $x\, is\, equivalent to\, y$.

Definition 2.13. *Let* $\theta_{rs} = \{(m_r, n_s)\}$ *be a double lacunary sequence; the two double sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are said to be strong asymptotically double lacunary equivalent of multiple L provided that*

$$\lim_{r,s} \tfrac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} \left| \tfrac{x_{mn}}{y_{mn}} - L \right| = 0,$$

that is x is equivalent to y and it is denoted by $N^{L}_{\theta_{rs}}$ *and simply strong asymptotically double lacunary equivalent if* $L = 1$. *In addition, let* $N^{L}_{\theta_{rs}}$ *denote the set of all sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *such that x is equivalent to y.*

Definition 2.14. *The double sequence* $\theta_{rs} = \{(m_r, n_s)\}$ *is called double lacunary sequence if there exist two increasing of integers such that*
$m_\circ = 0, h_r = m_r - m_{r-1} \to \infty$ *as* $r \to \infty$ *and* $n_\circ = 0, \bar{h}_s = n_s - n_{s-1} \to \infty$ *as* $s \to \infty$.
Notations: $m_{r,s} = m_r m_s, h_{r,s} = h_r \bar{h}_s$ *and* θ_{rs} *is determined by*
$I_{rs} = \{(m,n) : m_{r-1} < m \leq m_r\, and\, n_{s-1} < n \leq n_s\}, q_r = \frac{m_r}{m_{r-1}}, \bar{q}_s = \frac{n_s}{n_{s-1}}$ *and* $q_{rs} = q_r \bar{q}_s$.

Definition 2.15. *Let* $\widetilde{M}$ *be an Musielak modulus function. The two non-negative double sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are said to be strong* $\widetilde{M}-$ *asymptotically double equivalent of multiple* $\bar{0}$ *provided that* $\left[\chi^{2F}_{\widetilde{M}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = F_\mu(x) = \lim_{k,\ell\to\infty} \frac{1}{k\ell}$

$$\left\{ \sum_{m=1}^{k} \sum_{n=1}^{\ell} \left[\widetilde{M} \left(f^{mn}_{k\ell} \left((m+n)! \left| \tfrac{x_{mn}}{y_{mn}} - \bar{0} \right| \right)^{(1/m)+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right] = 0 \right\} = \lim_{k,\ell\to\infty} \tfrac{1}{k\ell}$$
$$\left\{ \tfrac{1}{f_{(k+2)(\ell+2)^{-1}}} \sum_{m=1}^{k} \sum_{n=1}^{\ell} \left[\widetilde{M} \left(f^{mn}_{k\ell} \left((m+n)! \left| \tfrac{x_{mn}}{y_{mn}} - \bar{0} \right| \right)^{(1/m)+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right] = 0 \right\},$$

$(k, \ell \in \mathbb{N})$, *and*

$$\left[\Lambda^{2F}_{\widetilde{M}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = F_\eta(x)$$
$$= \sup_{k,\ell} \frac{1}{k\ell} \left\{ \sum_{m=1}^{k} \sum_{n=1}^{\ell} \left[\widetilde{M} \left(f^{mn}_{k\ell} \left(\left| \frac{x_{mn}}{y_{mn}} - \bar{0} \right| \right)^{(1/m)+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right] < \infty \right\}$$
$$= \sup_{k,\ell} \tfrac{1}{k\ell}$$
$$\left\{ \tfrac{1}{f_{(k+2)(\ell+2)^{-1}}} \sum_{m=1}^{k} \sum_{n=1}^{\ell} \left[\widetilde{M} \left(f^{mn}_{k\ell} \left((m+n)! \left| \tfrac{x_{mn}}{y_{mn}} - \bar{0} \right| \right)^{(1/m)+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right] < \infty \right\},$$

$(k, \ell \in \mathbb{N})$, *it is denoted by* $\left[\widetilde{M}\right]^{\bar{0}}$ *and simply* $\widetilde{M}-$ *asymptotically double.*

Definition 2.16. *Let* $\widetilde{M}$ *be an Musielak modulus function and* $\theta_{rs} = \{(m_r, n_s)\}$ *be a double lacunary sequence; the two double sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are said to be strong* $\widetilde{M}-$ *asymptotically double lacunary of multiple* $\bar{0}$

$$\left[\chi^{2F}_{\widetilde{M}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = F_\mu(x)$$
$$= \lim_{r,s\to\infty} \frac{1}{h_{r,s}}$$
$$\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[\widetilde{M}\left(f^{mn}_{k\ell}\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}-\bar{0}\right|\right)^{(1/m)+n}, \left\|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\|_p\right)\right)\right] = 0\right\}.$$
$$= \lim_{r,s\to\infty} \frac{1}{h_{r,s}}$$
$$\left\{\frac{1}{f_{(k+2)(\ell+2)-1}}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[\widetilde{M}\left(f^{mn}_{k\ell}\left(\left|\frac{x_{mn}}{y_{mn}}-\bar{0}\right|\right)^{(1/m)+n}, \left\|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\|_p\right)\right)\right] = 0\right\},$$

$(k,\ell\in\mathbb{N})$, *and*

$$\left[\Lambda^{2F}_{\widetilde{M}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = F_\eta(x)$$
$$= \sup_{r,s} \frac{1}{h_{r,s}}$$
$$\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[\widetilde{M}\left(f^{mn}_{k\ell}\left(\left|\frac{x_{mn}}{y_{mn}}-\bar{0}\right|\right)^{(1/m)+n}, \left\|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\|_p\right)\right)\right] < \infty\right\} = \sup_{r,s} \frac{1}{h_{r,s}}$$
$$\left\{\frac{1}{f_{(k+2)(\ell+2)-1}}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[\widetilde{M}\left(f^{mn}_{k\ell}\left(\left|\frac{x_{mn}}{y_{mn}}-\bar{0}\right|\right)^{(1/m)+n}, \left\|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\|_p\right)\right)\right] < \infty\right\},$$

$(k,\ell\in\mathbb{N})$, *provided that is denoted by* $N_{\theta_{r,s}}^{\left[\widetilde{M}\right]^{\bar{0}}}$ *and simply strong* $\widetilde{M}-$ *asymptotically double lacunary.*

Consider the metric space $\left[\Lambda^{2F}_{\widetilde{M}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *with the metric*

$$d(x,y) = \sup_{k\ell}\left\{\widetilde{M}(F_\eta(x) - F_\eta(y)) : m,n = 1,2,3,\cdots\right\}. \tag{7}$$

Consider the metric space $\left[\chi^{2F}_{\widetilde{M}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *with the metric*

$$d(x,y) = \sup_{k\ell}\left\{\widetilde{M}(F_\mu(x) - F_\mu(y)) : m,n = 1,2,3,\cdots\right\}. \tag{8}$$

3. Almost asymptotically lacunary convergence of $PP-$ spaces

The idea of statistical convergence was first introduced by Steinhaus in 1951 and then studied by various authors. In this paper has studied the concept of statistical convergence in probabilistic $p-$ metric spaces.

Definition 3.1. *A triple* $(X,P,*)$ *be a* $PP-$ *space. Then a sequence* $x = (x_{mn})$ *is said to statistically convergent to* $\bar{0}$ *with respect to the probabilistic* $p-$ *metric* $P-$ *provided that for every* $\epsilon > 0$ *and* $\gamma \in (0,1)$

$$\delta\left(\left\{m,n\in\mathbb{N} : P_{((m+n)!|x_{mn}|)^{1/m+n}}(\epsilon) \leq 1-\gamma\right\}\right) = 0$$

or equivalently

$$\lim_{k\ell}\frac{1}{k\ell}\left|\left\{m\leq k, n\leq \ell : P_{((m+n)!|x_{mn}|)^{1/m+n}}(\epsilon) \leq 1-\gamma\right\}\right| = 0$$

In this case we write $St_P - lim x = \bar{0}$.

Definition 3.2. *A triple* $(X,P,*)$ *be a* $PP-$ *space. The two non-negative sequences* $x = (x_{mn})$ *and* $y = (y_{mn})$ *are said to be almost asymptotically statistical equivalent of multiple* $\bar{0}$ *in* $PP-$ *space* X *if for every* $\epsilon > 0$ *and* $\gamma \in (0,1)$.

$$\delta\left(\left\{m,n\in\mathbb{N}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right)=0$$

or equivalently

$$\lim_{k\ell}\tfrac{1}{k\ell}\left|\left\{m\leq k,n\leq \ell:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=0.$$

In this case we write $x\overset{\widehat{S}(PP)}{\equiv}y$.

Definition 3.3. *A triple* $(X,P,*)$ *be a* $PP-$ *space and* $\theta=(m_rn_s)$ *be a lacunary sequence. The two non-negative sequences* $x=(x_{mn})$ *and* $y=(y_{mn})$ *are said to be a almost asymptotically lacunary statistical equivalent of multiple* $\bar{0}$ *in* $PP-$ *space* X *if for every* $\epsilon>0$ *and* $\gamma\in(0,1)$

$$\delta_\theta\left(\left\{m,n\in I_{r,s}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right)=0 \tag{9}$$

or equivalently

$$\lim_{rs}\tfrac{1}{h_{rs}}\left|\left\{m,n\in I_{rs}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=0.$$

In this case we write $x\overset{\widehat{S_\theta}(PP)}{\equiv}y$.

Lemma 3.1. *A triple* $(X,P,*)$ *be a* $PP-$ *space. Then for every* $\epsilon>0$ *and* $\gamma\in(0,1)$, *the following statements are equivalent:*

(1) $\lim_{rs}\tfrac{1}{h_{rs}}\left|\left\{m,n\in I_{rs}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=0,$

(2) $\delta_\theta\left(\left\{m,n\in I_{r,s}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right)=0,$

(3) $\delta_\theta\left(\left\{m,n\in I_{r,s}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right)=1,$

(4) $\lim_{rs}\tfrac{1}{h_{rs}}\left|\left\{m,n\in I_{rs}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=1.$

4. Main results

Theorem 4.1. *A triple* $(X,P,*)$ *be a* $PP-$ *space. If two sequences* $x=(x_{mn})$ *and* $y=(y_{mn})$ *are almost asympototically lacunary statistical equivalent of multiple* $\bar{0}$ *with respect to the probabilistic* $p-$ *metric* P, *then* $\bar{0}$ *is unique sequence.*

Proof. Assume that $x\overset{\widehat{S_\theta^{\bar{0}}}(PP)}{\equiv}y$. For a given $\lambda>0$ choose $\gamma\in(0,1)$ such that $(1-\gamma)>1-\lambda$. Then, for any $\epsilon>0$, define the following set:

$$K=\left\{m,n\in I_{r,s}:P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}.$$

Then, clearly

$$\lim_{rs}\frac{K\bigcap\bar{0}}{h_{rs}}=1,$$

so K is non-empty set,since $x \overset{\widehat{S^{\bar{0}}_{\theta}}(PP)}{\equiv} y, \delta_\theta(K) = 0$ for all $\epsilon > 0$, which implies $\delta_{theta}(\mathbb{N} - K) = 1$. If $m, n \in \mathbb{N} - K$, then we have

$$P_{\bar{0}}(\epsilon) = P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > (1-\gamma) \geq 1 - \lambda$$

since λ is arbitrary, we get $P_{\bar{0}}(\epsilon) = 1$.

This completes the proof. □

Theorem 4.2. *A triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s)$, $\widehat{S_\theta}(PP) \subset \widehat{S}(PP)$ if $lim \sup_{rs} q_{rs} < \infty$.*

Proof. If $lim \sup_{rs} q_{rs} < \infty$. then there exists a $B > 0$ such that $q_{rs} < B$ for all $r, s \geq 1$. Let $x \overset{\widehat{S_\theta}(PP)}{\equiv} y$ and $\epsilon > 0$. Now we have to prove $\widehat{S}(PP)$. Set

$$K_{rs} = \left|\left\{m, n \in I_{r,s} : P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > 1 - \gamma\right\}\right|.$$

Then by definition, for given $\epsilon > 0$, there exists $r_0 s_0 \in \mathbb{N}$ such that

$$\frac{K_{rs}}{h_{rs}} < \frac{\epsilon}{2B} \text{ for all } r > r_0 \text{ and } s > s_0.$$

Let $M = max\{K_{rs} : 1 \leq r \leq r_0, 1 \leq s \leq s_0\}$ and let uv be any positive integer with $m_{r-1} < u \leq m_r$ and $n_{s-1} < v \leq n_s$. Then

$$\frac{1}{uv}\left|\left\{m \leq u, n \leq v : P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > 1 - \gamma\right\}\right| \leq$$
$$\frac{1}{m_{r-1}n_{s-1}}\left|\left\{m \leq m_r, n \leq n_s : P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > 1 - \gamma\right\}\right| = \frac{1}{m_{r-1}n_{s-1}}\{K_{11} + \cdots K_{rs}\}$$
$$\leq \frac{M}{m_{r-1}n_{s-1}} r_0 s_0 + \frac{\epsilon}{2B} q_{rs} \leq \frac{M}{m_{r-1}n_{s-1}} r_0 s_0 + \frac{\epsilon}{2}.$$

This completes the proof. □

Theorem 4.3. *A triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s)$, $\widehat{S}(PP) \subset \widehat{S_\theta}(PP)$ if $lim \inf_{rs} q_{rs} > 1$.*

Proof. If $lim \inf_{rs} q_{rs} > 1$, then there exists a $\beta > 0$ such that $q_{rs} > 1 + \beta$ for sufficiently large rs, which implies

$$\frac{h_{rs}}{K_{rs}} \geq \frac{\beta}{1+\beta}.$$

Let $x \overset{\widehat{S^{\bar{0}}}(pp)}{\equiv} y$, then for every $\epsilon > 0$ and for sufficiently large r, s, we have

$$\frac{1}{m_r n_s}\left|\left\{m \leq m_r, n \leq n_s : P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > 1 - \gamma\right\}\right| \geq$$
$$\frac{1}{m_r n_s}\left|\left\{m, n \in I_{rs} : P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > 1 - \gamma\right\}\right| \geq$$
$$\frac{\beta}{1+\beta}\frac{1}{h_{rs}}\left|\left\{m, n \in I_{rs} : P_{\left((m+n)!\left|\frac{x_{mn}}{y_{mn}}\right|\right)^{1/m+n} - \bar{0}}(\epsilon) > 1 - \gamma\right\}\right|.$$

Therefore $x \overset{\widehat{S^{\bar{0}}\theta}(pp)}{\equiv} y$.

This completes the proof. □

Corollary 4.1. *A triple* $(X, P, *)$ *be a* $PP-$ *space. For any lacunary sequence* $\theta = (m_r n_s)$, *with* $1 < lim \inf_{rs} q_{rs} \leq lim \sup_{rs} q_{rs} < \infty$, *then* $\widehat{S}(PP) = \widehat{S_\theta}(PP)$.

Proof. The result clearly follows from Theorem 4.2 and Theorem 4.3. □

5. Competing interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

6. Conclusion

This paper contains generalized results for the concept of almost asymptotically lacunary statistical χ^2 over probabilistic $p-$ metric spaces with some general topological properties. Researchers can extend the results for more general cases.

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